SOME PROPERTIES OF TWO DIMENSIONAL QUADRATIC MAPPINGS

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ABSTRACT

This paper has presented a thorough exploration of quadratic mappings within two-dimensional dynamical systems, employing a novel geometric approach known as graphical analysis. Through this method, we have gained valuable insights into the behavior of orbits, fixed points, periodic cycles, and bifurcation phenomena inherent in quadratic mappings. The graphical analysis procedure outlined in this paper offers an intuitive and visually appealing way to understand the dynamics of complex systems. By superimposing the graphs of relevant functions, we have been able to discern patterns, predict behaviors, and uncover the underlying structure of quadratic mappings

Keywords: *Mappings, attractors, basins, fixed point.*

Introduction

We introduce a geometric procedure that will help us understand the dynamics of two-dimensional mappings. This procedure, called graphical analysis, enables us to use the graphs of functions to determine the behavior of orbits in many cases. Suppose we have the two-dimensional mapping

$$Q_{c_1c_2} = \begin{cases} x' = f(y, c_1) \\ y' = f(x, c_2) \end{cases} \quad (1)$$

and wish to display the orbit of a given point (x_0, y_0). We begin by superimposing the graph of $x = f(y, c_1)$ on the graph of $y = f(x, c_2)$. The points of intersection of the graph $x = f(y, c_1)$ with the graph of $y = f(x, c_2)$ give us the fixed points of $Q_{c_1c_2}$. To find the orbit of (x_0, y_0), we begin at the point (x_0, y_0) on the XOY plane. We first draw a horizontal line to the graph of $x = f(y, c_1)$. When this line meets the graph of $x = f(y, c_1)$, we have reached the point (x', y_0) then draw a vertical line and denote it by V_1. We again begin at the point (x_0, y_0) on the XOY plane we draw a vertical line to the graph of $y = f(x, c_2)$. When this line meets the graph of $y = f(x, c_2)$, we have reached the point (x_0, y') then draw a horizontal line and denote it by H_1. The intersection point of V_1 and H_1 (x', y') is next point of the orbit of given point (x_0, y_0). To display the orbit of (x_0, y_0) geometrically, we thus continue this procedure over and over, in the next step we denote V_{i+1} instead of V_i and H_{i+1} instead of H_i. The intersection point of V_i and H_i is the i th point of the orbit of

(x_0, y_0) by the mapping of $Q_{c_1c_2}$. In the Fig. 1. we depicted graphical analysis of (x_0, y_0) by $Q_{c_1c_2} = \begin{cases} x' = y^2 + c_1 \\ y' = x^2 + c_2. \end{cases}$

Main part

It is tame that $Q_{c_1c_2}$quadratic mapping maps the vertical interval to the horizontal interval and inversely. Interval may be contract or extend. This proved the following theorem.

Theorem 1. There are may be only one fixed point on one horizontal and one vertical line for $Q_{c_1c_2} = \begin{cases} x' = f(y, c_1) \\ y' = f(x, c_2) \end{cases}$.

$Q_{c_1c_2}$quadratic mapping maps the rectangles to the rectangles, length and wide may be contract or extend.

Let the graph of $x = f(y, c_1)$ tangents with the graph of $y = f(x, c_2)$ externally and denote tangent point by (a, b) Fig. 2. Then this point is fixed point for $Q_{c_1c_2}$.

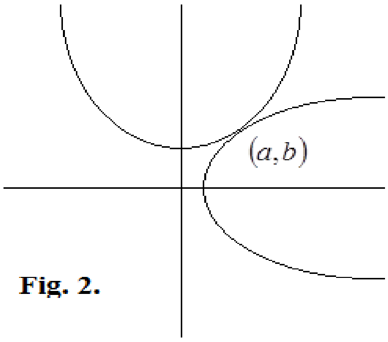
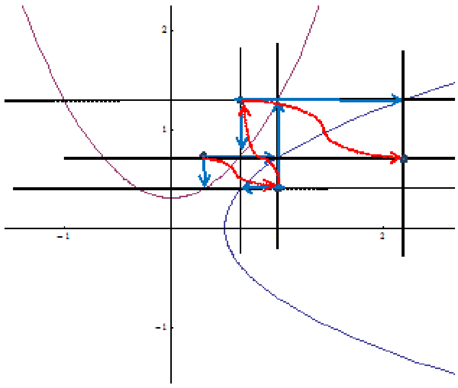


Fig. 2.



Corollary: Graphical analysis shows that the points does not belong to rectangle with vertices $(\pm a, \pm b)$ *are tend to infinite. The points belong to rectangle with vertices* $(\pm a, \pm b)$ *are tend to* (a, b). *It means the set all points belong to rectangle with vertices* $(\pm a, \pm b)$ *is the filled Julia set for* $Q_{c_1c_2}$.

Now let us the graph of $x = f(y, c_1)$ intersects with the graph of $y = f(x, c_2)$ on two points Fig.4. When c_1, c_2 parameters change the position of graphs of $x = f(y, c_1)$ and $y = f(x, c_2)$ functions change from tangent to intersection. There are two types of bifurcation occur in one time at tangent point. One is saddle – node bifurcation Fig. 3.

a. and one is period doubling bifurcation. Fig. 3.b. Fixed point (a,b) becomes repelling and there appears an attracting fixed point (a_1,b_1). This is saddle – node bifurcation. But the Julia set does not change.

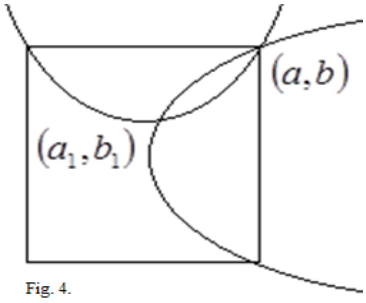
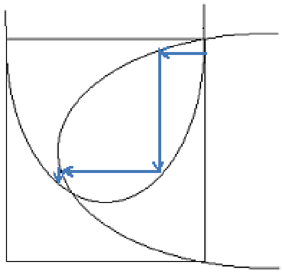


Fig. 4.



It is easy to see $(a,b_1)\xrightarrow{Q_{c_1c_2}}(a_1,b)\xrightarrow{Q_{c_1c_2}}(a,b_1)$ are periodic points with period two. This is period doubling bifurcation. Graphical analysis shows that every two fixed points have one periodic cycle with period two. It proves the following theorem.

Theorem 2. We have $Q_{c_1c_2}=\begin{cases}x'=f(y,c_1)\\ y'=f(x,c_2)\end{cases}$ *mapping. If* $Q_{c_1c_2}^{(2k-1)}, k\in N$ *mapping has* n *fixed points then it has also* $n(n-1)/2$ *periodic cycles with period two.*

Indeed n fixed points has different $n(n-1)/2$ pair points.

Graphical analysis shows that following corollary is true.

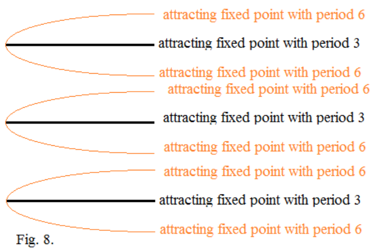


Fig. 8.

Corollary. Orbits of every points by $Q_{c_1c_2}=\begin{cases}x'=y^2+c_1\\ y'=x^2+c_2\end{cases}$ *in the Julia set, does not go out from the box* $c_1\le x\le\sqrt{c_1-c_2}, c_2\le y\le c_2^2+c_2$.

We denote the greater fixed point of $Q_{c_1c_2}$ by (a,b) then other fixed points.

$$\begin{cases}a=b^2+c_1\\ b=a^2+c_2\end{cases}\Rightarrow\begin{cases}c_1=a-b^2\\ c_2=b-a^2\end{cases}\Rightarrow Q_{ab}=\begin{cases}x'=y^2+a-b^2\\ y'=x^2+b-a^2.\end{cases}$$

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At the time of bifurcation eigenvalues equal one in absolute value

$$|\lambda_{1,2}| = 2\sqrt{ab} = 1.$$

When $4ab < 1$ fixed point not,
When $4ab = 1$ one neutral fixed point,
When $4ab > 1$ two fixed point one (a, b) is change to repelling and one (a_1, b_1) is attracting fixed point which born from (a, b) and satisfies $4a_1b_1 < 1$ inequality.
The area between hyperbolas $4ab > 1$ and $4a_1b_1 < 1$ is fixed point area in Mandelbrot set.
When $4a_1b_1 = 1$ occurs interesting type of bifurcation, periodic cycle with period four birth from (a_1, b_1). (a_1, b_1) changes from attracting to repelling and attracting periodic cycle with period four born. I called this type of *bifurcation quard – bifurcation.* Fig. 6.

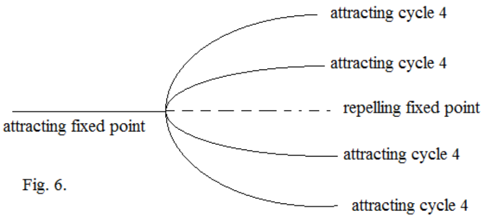

Fig. 6.

Let the points $(p_1, q_1), (p_2, q_2), (p_3, q_3) and (p_4, q_4)$ are periodic with period four born from (a_1, b_1) which $q_2 = q_1, q_3 = q_4, p_1 = p_4, p_2 = p_3$ multiplier of it $16\sqrt{p_1q_1p_2q_2p_3q_3p_4q_4} = 16p_1q_1p_2q_2$. The area between hyperbolas $4a_1b_1 > 1$ and $16p_1q_1p_2q_2 < 1$ is periodic point area with period four in Mandelbrot set.
When the multiplier $16p_1q_1p_2q_2 = 1$ occur two period doubling bifurcation at one moment, attracting periodic points with period four changes to repelling and born of then two attracting periodic points with period eight Fig. 7.

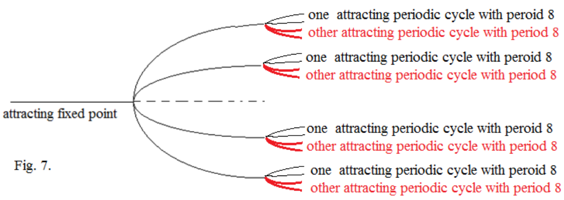

Fig. 7.

Continue this fashion in the next steps born four attracting cycle with period 16, then born eight attracting cycle with period 32, in general born 2^{n-2} attracting cycle with period 2^n.
Theorem 3. If Q_{ab} quadratic operator has period three, it also has period six.
Proof.
This time occur pitchfork bifurcation Fig. 8.

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Let Q_{ab} has period three, operator $Q_{ab}^{(3)}$ has at least three fixed point. By theorem 1. $Q_{ab}^{(3)}$ has at least three cycles with period two. Then $Q_{ab}^{(6)} = \left(Q_{ab}^{(3)}\right)^{(2)}$ has fixed point. If $Q_{ab}^{(6)}$ has fixed point Q_{ab} has period six. Q.E.D.

Definition: The filled-in Julia set for quadratic operator Q_{ab} is

$K_{Q_{ab}} = \left\{X \in R^2 | theorbit \left\{Q_{ab}^{(n)}(X)\right\}_{n=0}^{\infty} isbounded\right\}$

Then its boundary coincides with the Julia set: $J_{Q_{ab}} = \partial K_{Q_{ab}}$

Definition: The critical points for quadratic operator Q_{ab} are all (x_c, y_c) which determinant of Jacobian matrix at these points equal zero $\Delta\left(J(Q_{ab}(x_c, y_c))\right) = 0$.

Definition: The basin of attraction point (x_p, y_p) of quadratic operator Q_{ab} is the set of points (x, y) such that $\|Q_{ab}^n(x, y) - Q_{ab}^n(x_p, y_p)| \to 0$ as $n \to \infty$.

Definition: The Mandelbrot set $M_{Q_{ab}}$ for quadratic operator Q_{ab} is the set of points in (a,b) parameter space, the union of basins of all attraction critic points of Q_{ab}.

When the bottoms of parabolas belong to *rectangle with vertices* $(\pm a, \pm b)$ the Julia set is totally connected set and this rectangle. If one of parabola's bottom does not belong to $(\pm a, \pm b)$ rectangle the Julia set changes from totally connected set $b - a^2 \geq -banda - b^2 \geq -a \Rightarrow a^2 \leq 2b, b^2 \leq 2a$. Let the bottom of $x = y^2 + a - b^2$ is out from $(\pm a, \pm b)$ rectangle $a^2 > 2b$. But the bottom of $y = x^2 + b - a^2$ belongs to $(\pm a, \pm b)$ rectangle in this case $b^2 \leq 2a$ Fig. 5.

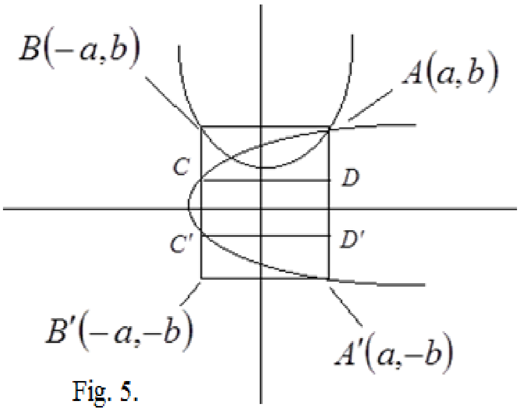


Fig. 5.

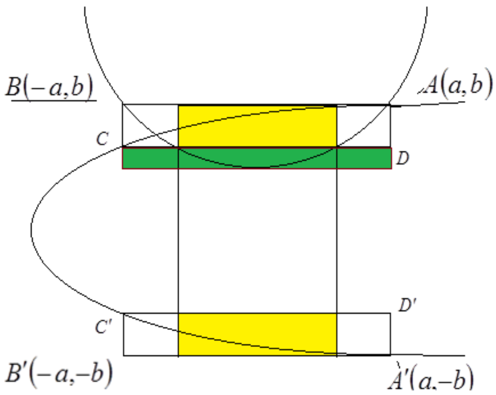


Fig. 6.

It is easy to find the coordinates of C, C', D, D'. $C(-a, \sqrt{b^2-2a}), C'(-a, -\sqrt{b^2-2a}), D(a, \sqrt{b^2-2a}), D'(a, -\sqrt{b^2-2a})$. If the bottom of $y = x^2 + b - a^2$ belongs to $ABCD$ then the Julia set is consist of $ABCD and A'B'C'D'$ disconnected rectangles. $A'B'C'D'$ is the pre-image of $ABCD$. The orbit of every point in $ABCD$ leaves in $ABCD$. The orbit of every point in $A'B'C'D'$ jumps to in $ABCD$ at the first iteration. The set of points below CD and above $C'D'$ go out from $ABCD$ at one iteration and tend to infinity. The set of such all points which go out from $ABCD$ at one iteration we denote by M_1, $Q_{ab}(M_1) \not\subset ABCD$. Let the bottom of $y = x^2 + b - a^2$ is below CD. Fig. 6. It is tame by graphical analysis all point belong to yellow rectangles jump to green rectangle at one iteration and at second iteration go out from $ABCD$. The vertical ends of yellow rectangles do not go out from $ABCD$. The set of such all points which go out from $ABCD$ at two iterations we denote by M_2, $Q_{ab}(M_2) \subset M_1$. We continue such fashion the set of such all points which go out from $ABCD$ at tree iterations we denote by M_3, $Q_{ab}(M_3) \subset M_2$. M_n is the set of such all points which go out from $ABCD$ at n iterations $Q_{ab}(M_n) \subset M_{n-1}$. Are there any points left after we throw out all of these rectangles from $ABCD$?

The ends of rectangles and the fixed points, eventually fixed points, periodic points, eventually periodic points left in $ABCD$? The set of all points left in $ABCD$ forever is Cantor type set we denote in by Λ

$$\Lambda = \{|x| \leq a, |y| \leq b\} \backslash \sum_{i=1}^{\infty} M_i$$

The $b = a^2/2, a = b^2/2$, $a^4 - 2a^2b + 2a = 0$ curves mean some boundaries of Mandelbrot set. We know from [1]

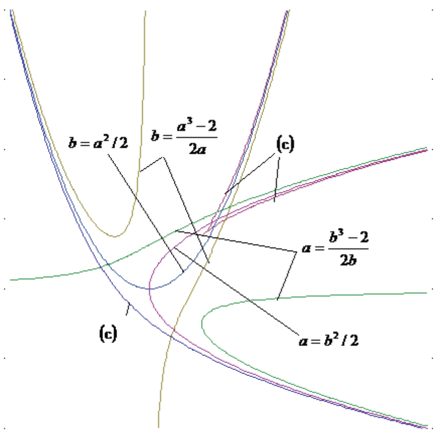
$4ab + 12(a+b) - 2\sqrt{(2a+2b+3)^3} + 9 = 0 and 4ab + 12(a+b) + 2\sqrt{(2a+2b+3)^3} + 9 = 0$ curves some boundaries of Mandelbrot set.

Conjecture: The closed set bounded with curves which graphics of following implicit or tacit functions on $R^2 = (a,b)$ parameter space

(a) $b = a^2/2 \text{ and } a = b^2/2$. (b) $b = \frac{a^3-2}{2a} and a = \frac{b^3-2}{2b}$

(c) $4ab = 1$

coincides with Mandelbrot set for quadratic operator Q_{ab}. It depicted here



Conclusion

In conclusion, this paper has presented a thorough exploration of quadratic mappings within two-dimensional dynamical systems, employing a novel geometric approach known as graphical analysis. Through this method, we have gained valuable insights into the behavior of orbits, fixed points, periodic cycles, and bifurcation phenomena inherent in quadratic mappings. The graphical analysis procedure outlined in this paper offers an intuitive and visually appealing way to understand the dynamics of complex systems. By superimposing the graphs of relevant functions, we have been able to discern patterns, predict behaviors, and uncover the underlying structure of quadratic mappings.

REFERENCES

[1] Seytov, Sh.J., Narziyev, N.B., Eshniyozov, A.I., Nishonov, S.N. The algorithms for developing computer programs for the sets of Julia and Mandelbrot. *AIP Conference Proceedings*, 2023, 2789, 050021.

[2] Seytov, S.J., Eshniyozov, A.I., Narziyev, N.B. Bifurcation Diagram for Two Dimensional Logistic Mapping. *AIP Conference Proceeding*, 2023, 2781, 020076.

[3] Seytov, S.J., Nishonov, S.N., Narziyev, N.B. Dynamics of the Populations Depend on Previous Two Steps. *AIP Conference Proceedings*., 2023, 2781, 020071.

[4] Seytov, S.J., Sayfullayev, B.Sh., Anorbayev, M.M. Separating a Finite System of Points from each Other in Real Euclidean Space *AIP Conference Proceedings*., 2023, 2781, 020043.

[5] Seytov, S.J., Eshmamatova, D.B. Discrete Dynamical Systems of Lotka–Volterra and Their Applications on the Modeling of the Biogen Cycle in Ecosystem. *Lobachevskii Journal of Mathematics*., 2023, 44(4), pp. 1471–1485.

[6] Eshmamatova, D.B., Seytov, S.J., Narziev, N.B. Basins of Fixed Points for Composition of the Lotka–Volterra Mappings and Their Classification. *Lobachevskii Journal of Mathematics*., 2023, 44(2), pp. 558–569

[7] Ganikhodzhaev, R.N., Seytov, Sh.J. Coexistence chaotic behavior on the evolution of populations of the biological systems modeling by three dimensional quadratic mappings *Global and Stochastic Analysis*., 2021, 8(3), pp 41–45.

[8] Ganikhodzhayev, R., Seytov, S. An analytical description of mandelbrot and Julia sets for some multi-dimensional cubic mappings/ *AIP Conference Proceedings*., 2021, 2365, 050006.