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ROLLING WITHOUT SLIPPING AND MANNHEIM CORRESPONDENCE OF CURVES ON S^2

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ABSTRACT

This paper investigates the relationships between two curves rolling over each other without slipping on the unit sphere in three-dimensional Euclidean space, together with their Mannheim partner curves, from the viewpoint of geodesic curvature. The corresponding relations between their geodesic Frenet frames are first derived, followed by an analysis of the relations between their geodesic curvatures.

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1. Introduction

Curves are one of the two main subjects studied in classical differential geometry. In such investigations, the properties of curves are examined through their components in a neighborhood of a point (Carmo, 1976). The Frenet frame, curvature, and torsion are used to characterize curves at a given point. The relative positions of two curves can be analyzed in terms of the elements of their Frenet frames. By establishing relations among the elements of the Frenet frames of two curves, mathematicians have defined several special pairs of associated curves. Among these, the *involute–evolute* pair of curves, whose tangent vectors at corresponding points are orthogonal (Huygens, 1673); the *Bertrand* pair of curves, which share a common principal normal vector (Bertrand, 1850); and the *Mannheim* pair of curves, in which the principal normal vector of one curve corresponds to the binormal vector of the other (Liu & Wang, 2008; Orbay & Kasap, 2009), are the most widely studied.

Mannheim curves on the sphere can be regarded as the adaptation of classical Mannheim pairs to the differential structure of spherical geometry. Şenyurt, Altun, and Cevahir (2017) provided a detailed analysis of Mannheim curve pairs by employing the Darboux vector together with the special Smarandache curves associated with the Sabban frame constructed on the sphere. Wang and Yuan (2020) proposed a method for generating Mannheim curves from spherical curves and demonstrated that all Mannheim curves can be obtained in this manner. Takahashi (2022) established the necessary conditions under which both regular and singular spherical curves in the three-dimensional sphere become Mannheim curves. Collectively, these studies indicate that Mannheim curves possess a structurally rich nature within the framework of spherical differential geometry.

The study of how pairs of curves transform under motion geometry is another intriguing topic. This subject is of interest not only for purely mathematical investigations but also for various applied fields such as physics, engineering, computer-aided design, robotics, and industrial manufacturing, where the motion of curves or even the joint motion of a curve and a surface play a significant role. In computer-aided design and manufacturing, offset curves and surfaces have been extensively examined both in terms of their geometric definitions and computational methods, and the fundamental concepts and algorithms in this field have been thoroughly discussed in the literature (Pham, 1992; Maekawa, 1999). In industrial manufacturing, problems such as tool-path generation, collision analysis, and surface quality optimization are also solved using algorithms based on surface kinematics (Lu, Zhang, Tian, Han & Wang, H. 2021; Qin, Jia, Ma, Ren, & Song, 2017). From a physical perspective, the motion of curves and surfaces has been studied in connection with fluid dynamics, materials science, and geometric evolution equations (Shi, Chen, & Xie, 2017), offering new insights into practical engineering problems (Wu, Wang, Wei, & Zhu, 2022). These studies demonstrate that spatial kinematic differential geometry is not merely a theoretical branch of mathematics, but also plays an important role in mechanism analysis, robot kinematics, and engineering applications.

The kinematic differential geometry of curves under spatial motion provides a powerful framework for examining the motion of curves from a geometric perspective. In this field, not only differential geometric concepts such as line motions, axoids, curvature, and torsion are employed, but also algebraic tools such as dual numbers and the Study transformation. For example, Wang, Liu, and Xiao (1997a, 1997b, 1997c) investigated the trajectories generated by a point and a line in spatial kinematics from a new viewpoint, determining the invariants of axoids and revealing their kinematic significance. Lan, Huijun, and Liuming (2002) decomposed plane pairs to explain the formation of curve pairs; by considering the distribution of these curve pairs, they divided the plane pairs into four distinct regions and defined and analyzed their limiting cases. Orbay and Şahin (2022) clarified the behavior of Mannheim offsets by utilizing Mannheim partner curves of two spherical curves under a one-parameter helical motion. Alluhaibi and Abdel-Baky (2023) examined the kinematic properties of special line trajectories and demonstrated their relations with the Disteli axis. Almoneef and Abdel-Baky (2023) investigated line motions in space within the framework of kinematic differential geometry, establishing spatial counterparts of the Euler-Savary and Disteli formulas. Building on this line of research, the kinematic geometry of line congruences in Euclidean 3-space was further analyzed, leading to the derivation of curvature relations and Hamilton–Mannheim formulas that extend the geometric interpretation of line motions (Almoneef & Abdel-Baky, 2025).

In this study, Mannheim partner curves corresponding to each other under the without slipping rolling motion on the unit sphere in three-dimensional Euclidean space are investigated. First, the relations between the geodesic Frenet frames of the Mannheim partner curves are derived. Then, the relationships between their geodesic curvatures are examined. Finally, an illustrative example is presented to support the theoretical results.

2. Preliminaries

Let two curves C and $\bar{C}$ roll over each other without slipping on the unit sphere S^2.
The curves C and $\bar{C}$ are parameterized by arc lengths s and $\bar{s}$, respectively. Let C^* and $\bar{C}^*$ denote the Mannheim partner curves of C and $\bar{C}$, parameterized by s^* and $\bar{s}^*$, respectively. The geodesic Frenet frames at the corresponding points $c(s)$, $c^*(s^*)$, $\bar{c}(\bar{s})$, and $\bar{c}^*(\bar{s}^*)$ of the Mannheim pairs (C, C^*) and $(\bar{C}, \bar{C}^*)$ are denoted by $\{e, t, g\}$, $\{e^*, t^*, g^*\}$, $\{\bar{e}, \bar{t}, \bar{g}\}$, and $\{\bar{e}^*, \bar{t}^*, \bar{g}^*\}$, respectively. For a curve lying on the unit sphere, let $\{e, t, g\}$denote the geodesic Frenet frame, where e is the position vector, t is the unit tangent vector, and $g = e \times t$ is the binormal vector. The derivatives of these frame vectors with respect to the arc-length parameter s satisfy

$$e_s = t, t_s = -e + \gamma g, g_s = -\gamma t, \qquad (1)$$

where γ denotes the geodesic curvature (Ravani & Ku, 1991). Similarly, the derivative equations of the geodesic Frenet frames $\{e^*, t^*, g^*\}$, $\{\bar{e}, \bar{t}, \bar{g}\}$, and $\{\bar{e}^*, \bar{t}^*, \bar{g}^*\}$ are given by

$$\bar{e}_{\bar{s}} = \bar{t}, \bar{t}_{\bar{s}} = -\bar{e} + \bar{\gamma}\, \bar{g}, \bar{g}_{\bar{s}} = -\bar{\gamma}\, \bar{t}, \qquad (2)$$

$$e^*_{s^*} = t^*, t^*_{s^*} = -e^* + \gamma^* g^*, g^*_{s^*} = -\gamma^* t^*, \qquad (3)$$

$$\bar{e}^*_{\bar{s}^*} = \bar{t}^*, \bar{t}^*_{\bar{s}^*} = -\bar{e}^* + \bar{\gamma}^*\, \bar{g}^*, \bar{g}^*_{\bar{s}^*} = -\bar{\gamma}^*\, \bar{t}^*. \qquad (4)$$

Under the without slipping rolling condition, these frames are related through rotation transformations at the points of tangency. The corresponding relations between the geodesic Frenet frames are then established to express how the Mannheim partner curves during the without slipping rolling motion on S^2.

3. A Mannheim curve pair rolling without slipping on the sphere S^2

Let the geodesic Frenet frames of the curves C and $\bar{C}$ on the sphere S^2 at their common point M be denoted by $\{e, t, g\}$ and $\{\bar{e}, \bar{t}, \bar{g}\}$, respectively. Let C^* and $\bar{C}^*$ denote the Mannheim partner curves of C and $\bar{C}$, and with corresponding geodesic Frenet frames $\{e^*, t^*, g^*\}$ and $\{\bar{e}^*, \bar{t}^*, \bar{g}^*\}$, respectively. The Mannheim curve pairs (C, C^*) and $(\bar{C}, \bar{C}^*)$ satisfy the following relations:

$$\begin{aligned} e^* &= (\cos\theta)e + (\sin\theta)t \\ t^* &= g \\ g^* &= (\sin\theta)e - (\cos\theta)t \end{aligned} \tag{5}$$

and

$$\begin{aligned} \bar{e}^* &= (\cos\beta)\bar{e} + (\sin\beta)\bar{t} \\ \bar{t}^* &= \bar{g} \\ \bar{g}^* &= (\sin\beta)\bar{e} - (\cos\beta)\bar{t}. \end{aligned} \tag{6}$$

Here, $\theta = \theta(s^*)$ and $\beta = \beta(\bar{s}^*)$ denote, respectively, the angles between the position vectors e and e^*, and between $\bar{e}$ and $\bar{e}^*$.

If the curves C and $\bar{C}$ roll over each other without slipping, the geodesic Frenet frames satisfy the following relations:

$$\begin{aligned} \bar{e} &= e \\ \bar{t} &= (\cos\alpha)t + (\sin\alpha)g \\ \bar{g} &= -(\sin\alpha)t + (\cos\alpha)g \end{aligned} \tag{7}$$

where, $\alpha = \alpha(\bar{s})$ is the angle between the tangent vectors t and $\bar{t}$. If $\alpha = 0$, the curves are tangent in the same direction, and their frames coincide. The quantity $\alpha_{\bar{s}} = \frac{d\alpha}{d\bar{s}}$ represents the rotational velocity of the geodesic Frenet frame in the tangent plane during the rolling without slipping (Struik, 1988).

Under the rolling without slipping motion on the unit sphere S^2, the geodesic Frenet frames of the curves C and $\bar{C}$ are related through a rotation about the normal direction at their instantaneous point of contact. These rotational relations, previously established in Equation (7), make it possible to express the Frenet frames of the Mannheim partner curves (C, C^*) and $(\bar{C}, \bar{C}^*)$ in terms of one another. By combining the frame transformations associated with (i) the Mannheim pair structure and (ii) the rolling without slipping condition, explicit formulas are obtained that describe the correspondence between the frames at matched points on these curves. The following corollaries summarize these relations.

Corollary 1. Under the rolling without slipping motion, at the corresponding points of the curves C and $\bar{C}^*$ lying on the sphere S^2, the following relations hold between their geodesic Frenet frames:

$$\begin{aligned} \bar{e}^* &= (\cos\beta)e + (\sin\beta\cos\alpha)t + (\sin\beta\sin\alpha)g \\ \bar{t}^* &= -(\sin\alpha)t + (\cos\alpha)g \\ \bar{g}^* &= (\sin\beta)e - (\cos\beta\cos\alpha)t - (\cos\beta\sin\alpha)g. \end{aligned} \tag{8}$$

Corollary 2. Under the rolling without slipping motion, at the corresponding points of the curves $\bar{C}$ and C^* lying on the sphere S^2, the following relations hold between their geodesic Frenet frames:

$$\begin{aligned} e^* &= (\cos\theta)\bar{e} + (\sin\theta\cos\alpha)\bar{t} - (\sin\theta\sin\alpha)\bar{g} \\ t^* &= (\sin\alpha)\bar{t} + (\cos\alpha)\bar{g} \\ g^* &= (\sin\theta)\bar{e} - (\cos\theta\cos\alpha)\bar{t} + (\cos\theta\sin\alpha)\bar{g} \end{aligned} \tag{9}$$

Corollary 3. Under the rolling without slipping motion, at the corresponding points of the curves C^* and $\bar{C}^*$ lying on the sphere S^2, the following relations hold between their geodesic Frenet frames:

$$\begin{aligned} \bar{e}^* &= (\cos\beta\cos\theta + \sin\beta\cos\alpha\sin\theta)e^* + (\sin\beta\sin\alpha)t^* + (\cos\beta\sin\theta - \sin\beta\cos\alpha\cos\theta)g^* \\ \bar{t}^* &= -(\sin\alpha\sin\theta)e^* + (\cos\alpha)t^* + (\sin\alpha\cos\theta)\,g^* \\ \bar{g}^* &= (\sin\beta\cos\theta - \cos\beta\cos\alpha\sin\theta)e^* - (\cos\beta\sin\alpha)t^* + (\sin\beta\sin\theta + \cos\beta\cos\alpha\cos\theta)g^* \end{aligned} \tag{10}$$

With the frame correspondences established, attention now turns to the geodesic curvatures of the curves. The following theorems present the curvature identities that arise from the interaction between the Mannheim structure and the rolling motion.

Theorem 1. While the curves C and $\bar{C}$ lying on the sphere S^2 roll over each other without slipping, the geodesic curvatures of the curves C, C^*, $\bar{C}$, and $\bar{C}^*$ are given by:

a. $\gamma = \frac{s_s^*}{\sin\theta}$,

b. $\gamma^* = \cot\theta$,

c. $\bar{\gamma} = \frac{\bar{s}_{\bar{s}}^*}{\sin\beta}$,

d. $\bar{\gamma}^* = \cot\beta$.

Proof. a. Taking the derivative of the relation for e^* given in Equation (5) with respect to the parameter s^* yields

$$e^*_{s^*} = (-\sin\theta\theta_{s^*})e + (\cos\theta)e_{s^*} + (\cos\theta\theta_{s^*})t + (\sin\theta)t_{s^*}.$$

Substituting Equations (1), (3), and (5) into the derivatives yields

$$g = (-\sin\theta\theta_{s^*})e + (\cos\theta)s_{s^*}t + (\cos\theta\theta_{s^*})t + (\sin\theta)(-e + \gamma g)s_{s^*}.$$

Simplifying, one obtains

$$g = (-sin\theta\theta_{s^*} - sin\theta s_{s^*})e + (cos\theta)(s_{s^*} + \theta_{s^*})t + (sin\theta\gamma s_{s^*})g$$

Assuming $sin\theta \neq 0$, it follows that

$$\theta_{s^*} = -s_{s^*}, \ \gamma = \frac{s_s^*}{sin\theta}.$$

b. Taking the derivative of t^* in Equation (5) with respect to s^* and using Equation (1) gives

$$t^*_{s^*} = -\gamma s_{s^*} t.$$

On the other hand, using Equations (3) and (5) yields

$$t^*_{s^*} = (-cos\theta + \gamma^* sin\theta)e + (-sin\theta - \gamma^* cos\theta)t.$$

Comparing the coefficients and assuming $sin\,\theta \neq 0$, it follows that

$$\gamma^* = cot\theta.$$

c. Taking the derivative of $\bar{e}^*$ in Equation (6) with respect to $\bar{s}^*$ gives

$$\bar{e}^*_{\bar{s}^*} = (-sin\beta\beta_{\bar{s}^*})\bar{e} + (cos\beta)\bar{e}_{\bar{s}^*} + (cos\beta\beta_{\bar{s}^*})\bar{t} + (sin\beta)\bar{t}_{\bar{s}^*}).$$

Substituting Equations (2), (4), and (6) into the derivatives yields

$$\bar{g} = (-sin\beta\beta_{\bar{s}^*})\bar{e} + (cos\beta)\bar{t}\bar{s}_{\bar{s}^*} + (cos\beta\beta_{\bar{s}^*})\bar{t} + (sin\beta)(-\bar{e} + \bar{\gamma}\bar{g})\bar{s}_{\bar{s}^*}.$$

Simplifying, one obtains

$$\bar{g} = (-sin\beta\beta_{\bar{s}^*} - sin\beta\bar{s}_{\bar{s}^*})\bar{e} + (cos\beta\bar{s}_{\bar{s}^*} + cos\beta\beta_{\bar{s}^*})\bar{t} + (sin\beta\bar{\gamma}\bar{s}_{\bar{s}^*})\bar{g}.$$

Assuming $sin\,\beta \neq 0$, the following that

$$\beta_{\bar{s}^*} = -\bar{s}_{\bar{s}^*}, \ \bar{\gamma} = \frac{\bar{s}^*_{\bar{s}}}{sin\beta}.$$

d. Taking the derivative of $\bar{t}^*$ in Equation (6) with respect to $\bar{s}^*$ and considering Equation (2) gives

$$\bar{t}^*_{\bar{s}^*} = -\bar{\gamma}\bar{s}_{\bar{s}^*}\bar{t}.$$

Using Equations (4) and (6), one obtains

$$-\bar{e}^* + \bar{\gamma}^*\bar{g}^* = (sin\beta\bar{\gamma}^* - cos\beta)\bar{e} + (-cos\beta\bar{\gamma}^* - sin\beta)\bar{t}.$$

Comparing the coefficients and assuming $sin\,\beta \neq 0$, it follows that

$$\bar{\gamma}^* = cot\beta.$$

Theorem 2. While the curves C and $\bar{C}$ lying on the sphere S^2 roll over each other without slipping, the geodesic curvatures of the curves $C, C^*, \bar{C}, \bar{C}^*$ satisfy the following relations:

a. $\gamma^* = s_{s^*}\sqrt{\gamma^2 - (s_s^*)^2}$,

b. $\bar{\gamma} = \bar{s}_s(\gamma + \alpha_s)$,

c. $\bar{\gamma}^* = \mp\sqrt{(\bar{s}_{\bar{s}^*})^2(\bar{s}_s)^2(\gamma + \alpha_s)^2 - 1}$,

d. $\bar{\gamma}^* = \bar{s}_{\bar{s}^*}\sqrt{\bar{\gamma}^2 - 1}$.

Proof. a. From Theorem 1(b), it follows that

$$sin\theta = \frac{1}{\sqrt{(\gamma^*)^2+1}}.$$

From Theorem 1(a), $sin\,\theta$ can also be expressed as

$$sin\theta = \frac{s_s^*}{\gamma}.$$

Equating these two expressions gives

$$\frac{1}{\sqrt{(\gamma^*)^2+1}} = \frac{s_s^*}{\gamma}$$

which leads to

$$\gamma^* = s_{s^*}\sqrt{\gamma^2-(s_s^*)^2}.$$

b. Taking the derivative of the relation for $\bar{t}$ given in Equation (7) with respect to the parameter $\bar{s}$ yields

$$\bar{t}_{\bar{s}} = s_{\bar{s}}[(-cos\alpha)\alpha_s t + (cos\alpha)t_s + (cos\alpha)\alpha_s g + (sin\alpha)g_s].$$

Substituting Equations (1) and (2) into this expression gives

$$-\bar{e}+\bar{\gamma}\bar{g} = s_{\bar{s}}[(-cos\alpha)e + (-sin\alpha)(\alpha_s+\gamma)t + (cos\alpha)(\alpha_s+\gamma)g].$$

Assuming $sin\alpha \neq 0$, the following relations are obtained:

$$cos\alpha = \bar{s}_s,\ \gamma = \alpha_s,\ \bar{\gamma} = \bar{s}_s(\gamma+\alpha_s).$$

c. From Theorem 1(d), it follows that

$$\bar{\gamma}^* = \frac{cos\beta}{sin\beta} = \frac{\sqrt{1-(sin\beta)^2}}{sin\beta}$$

Using Theorem 1(c) and Theorem 2(b) and carrying out the required computations, one obtains

$$\bar{\gamma}^* = \frac{\sqrt{1-\left(\frac{\bar{s}_{\bar{s}}^*}{\bar{\gamma}}\right)^2}}{\frac{\bar{s}_{\bar{s}}^*}{\bar{\gamma}}},$$

$$\bar{\gamma}^* = \bar{s}_s(\gamma+\alpha_s)\bar{s}_{\bar{s}^*}\sqrt{1-\left(\frac{\bar{s}_{\bar{s}}^*}{\bar{s}_s(\gamma+\alpha_s)}\right)^2},$$

$$\bar{\gamma}^* = \mp\bar{s}_{\bar{s}^*}\sqrt{(\bar{s}_s)^2(\gamma+\alpha_s)^2-(\bar{s}_{\bar{s}}^*)^2},$$

$$\bar{\gamma}^* = \mp\sqrt{(\bar{s}_{\bar{s}^*})^2(\bar{s}_s)^2(\gamma+\alpha_s)^2-1}.$$

d. From Theorem 1(d), it follows that

$$\bar{\gamma}^* = \frac{cos\beta}{sin\beta} = \frac{\sqrt{1-(sin\beta)^2}}{sin\beta}$$

From Theorem 1(c), $sin\beta$ can also be expressed as

$$sin\beta = \frac{\bar{s}_{\bar{s}}^*}{\bar{\gamma}}$$

Equating these two expressions gives

$$\bar{\gamma}^* = \frac{\sqrt{1-(\frac{\bar{s}_{\bar{s}}^*}{\bar{\gamma}})^2}}{\frac{\bar{s}_{\bar{s}}^*}{\bar{\gamma}}}$$

$$\bar{\gamma}^* = \frac{\sqrt{\bar{\gamma}^2 - 1}}{\bar{s}^*_{\bar{s}}}$$

which leads to

$$\bar{\gamma}^* = \bar{s}_{\bar{s}^*}\sqrt{\bar{\gamma}^2 - 1}.$$

The following corollaries can be derived from Theorem 2.

Corollary 4. If the curve C is geodesic, then the geodesic curvature of the curve $\bar{C}$ depends on the rate of change of its tangent direction ($\bar{\gamma} = \bar{s}_s \alpha_s$).

Corollary 5. If the curve $\bar{C}$ is geodesic, then the geodesic curvature of the curve C depends on the rate of change of its tangent direction in the opposite direction ($\gamma = -\alpha_s$).

Corollary 6. If the curve C^* is geodesic, then the geodesic curvature of the curve C is equal to the ratio between the two parametrizations ($\gamma = s^*_s$).

Corollary 7. If the curve $\bar{C}^*$ is geodesic, then the geodesic curvature of the curve $\bar{C}$ is constant ($\bar{\gamma} = \mp 1$).

A clearer understanding of the above findings may be achieved by examining a concrete example in which some of these results can be explicitly observed.

Example 1. Let C and $\bar{C}$ be two curves on S^2 given by

$$C(s) = (cos\varphi cos(\frac{s}{\cos\varphi}),\ cos\varphi sin(\frac{s}{\cos\varphi}),\ sin\varphi),$$

$$\bar{C}(\bar{s}) = (cos\chi cos(\frac{\bar{s}}{cos\chi}), sin\chi,\ cos\chi sin(\frac{\bar{s}}{cos\chi})),$$

where φ and χ are constant angles with $\varphi \neq 0$ and $\chi \neq 0$.

First, consider the geodesic Frenet frame $\{e, t, g\}$ and the geodesic curvature γ of the curve C. We have

$$e = (cos\varphi cos(\frac{s}{\cos\varphi}),\ cos\varphi sin(\frac{s}{\cos\varphi}),\ sin\varphi),$$

$$t = (-sin(\frac{s}{\cos\varphi}), cos(\frac{s}{\cos\varphi}),\ 0),$$

$$g = (-sin\varphi cos(\frac{s}{\cos\varphi}), -sin\varphi sin(\frac{s}{\cos\varphi}), cos\varphi).$$

The geodesic curvature is

$$\gamma = \langle t_s, g\rangle,$$

$$\gamma = \langle\left(-\frac{1}{\cos\varphi}cos(\frac{s}{\cos\varphi}), -\frac{1}{\cos\varphi}sin(\frac{s}{\cos\varphi}), 0\right), \left(-sin\varphi\, cos(\frac{s}{\cos\varphi}), -sin\varphi sin(\frac{s}{\cos\varphi}), cos\varphi\right)\rangle,$$

$$\gamma = \frac{sin\varphi}{cos\varphi} = tan\varphi.$$

Thus, the geodesic curvature of C is constant.

Similarly, the geodesic Frenet frame $\{\bar{e}, \bar{t}, \bar{g}\}$ and the geodesic curvature $\bar{\gamma}$ of the curve $\bar{C}$ are given by

$$\bar{e} = (cos\chi cos(\frac{\bar{s}}{cos\chi}), sin\chi,\ cos\chi sin(\frac{\bar{s}}{cos\chi})),$$

$$\bar{t} = (-sin(\frac{\bar{s}}{cos\chi}), 0, cos(\frac{\bar{s}}{cos\chi})),$$

$$\bar{g} = (sin\chi\, cos\left(\frac{\bar{s}}{cos\chi}\right), cos\chi, sin\chi sin\left(\frac{\bar{s}}{cos\chi}\right)),$$

from which

$$\bar{\gamma} = -tan\chi.$$

Hence, the geodesic curvature of $\bar{C}$is also constant.

Under the condition of rolling without slipping, the equality $e = \bar{e}$ is imposed.

$$cos\varphi cos(\frac{s}{cos\,\varphi}) = cos\chi cos(\frac{\bar{s}}{cos\chi}),\;\; cos\varphi sin\left(\frac{s}{cos\,\varphi}\right) = sin\chi,\;\; sin\varphi = cos\chi sin(\frac{\bar{s}}{cos\chi}),$$

it follows that

$$\frac{sin\chi}{cos\varphi} = sin\left(\frac{s}{cos\varphi}\right),\;\; \frac{sin\varphi}{cos\chi} = sin(\frac{\bar{s}}{cos\chi}).$$

Differentiating the relation $e = \bar{e}$ yields

$$e_s s_{\bar{s}} = \bar{e}_{\bar{s}},\;\; t s_{\bar{s}} = \bar{t},$$

and therefore

$$s_{\bar{s}} = \langle t, \bar{t}\rangle = sin\left(\frac{s}{cos\varphi}\right) sin\left(\frac{\bar{s}}{cos\chi}\right) = tan\chi tan\varphi.$$

Thus,

$$s_{\bar{s}} = tan\varphi tan\chi = -\gamma\bar{\gamma}.$$

Now, using the relation

$$\bar{\gamma} = \bar{s}_s(\gamma + \alpha_s),$$

we obtain

$$\alpha_s = \bar{\gamma} s_{\bar{s}} - \gamma,$$

$$\alpha_s = -tan^2\chi tan\varphi - tan\varphi = -tan\varphi sec^2\chi.$$

This expression describes how the angle between the tangent vectors changes during the rolling motion without slipping.

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